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Noether-Charge Realization of Diffeomorphism Algebra

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Abstract. It is shown in the covariant phase space formalism that the Noether charges with respect to the diffeomorphism generated by vector fields and their horizontal variations in general relativity form a diffeomorphism algebra. It is also shown with the help of the null tetrad which is well defined everywhere that the central term of the reduced diffeomorphism algebra on the Killing horizon for a large class of vector fields vanishes.

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1 Introduction

It is well known that in the canonical formalism of general relativity (GR), the Hamiltonian constraint $\mathcal{H} \approx 0$ and 3-dimensional diffeomorphism constraint $\mathcal{H}_i \approx 0$ form an algebra under the Poisson bracket [1, 2], which reflects the diffeomorphism structure of space-time. However, the canonical approach does not preserve the covariance of GR manifestly. To recover the manifest covariance of the theory, a new approach, the covariant phase space formalism, has been developed [3]–[10].

In studying the statistical origin of the black hole entropy by use of the covariant phase space method [11], Carlip treated the Hamiltonian functional conjugate to a vector field the generator of the diffeomorphism algebra just like the Hamiltonian in the canonical approach. The diffeomorphism algebra is assumed to be realized by the Poisson bracket or by the Dirac bracket [12, 6] on the constraint surface. Unfortunately, the Hamiltonian functional

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conjugate to a vector field ξ^a does not always exist, as pointed out by Wald [8], for a given boundary condition. In addition, in the lack of the definition of the Poisson bracket and thus the Dirac bracket in the covariant phase space formalism, Carlip borrowed the Poisson bracket and the Dirac bracket from the ADM formalism [1, 2].

On the other hand, in both classical and quantum field theories, if a Lagrangian possesses certain symmetries, such as gauge symmetry and Poincaré symmetry, the corresponding Lie algebras can always be generated by the Noether charges of the conservation currents with respect to the symmetries. This spirit should also be available for the diffeomorphism invariance of a diffeomorphism invariant theory such as GR since the set of diffeomorphisms forms an infinite dimensional group under composition [13]. The main purpose of the present letter is to show how the algebra $\mathit{diff}(\mathcal{M})$ can be realized from the Noether charges in GR. It is also shown with the help of the null tetrad which is well defined everywhere that the central term of the reduced diffeomorphism algebra on the Killing horizon for a large class of vector fields vanishes.

The letter is arranged in the following way. In the next section, we briefly review the Noether currents with respect to the diffeomorphisms generated by vector fields and their charges in a diffeomorphism invariant theory. In section 3, we show that the diffeomorphism algebra may be realized by the Noether charges and their horizontal variations. In section 4, we study the reduced algebra on the event horizon of black-hole space-time manifold. In the last section, some remarks are given.

2 Noether Currents and Their Charges

Let $\mathbf{L}$ be the Lagrangian 4-form of a diffeomorphism-invariant gravitational metric theory. Its horizontal variation induced by the vector field ξ^a can be written as [8, 9, 14]

$$\hat{\delta}_\xi \mathbf{L} = \mathbf{E}\hat{\delta}_\xi g + d\mathbf{\Theta}(g, \hat{\delta}_\xi g), \tag{1}$$

where $\mathbf{E} = 0$ gives rise to the Euler-Lagrange equation for the theory and $\mathbf{\Theta}(g, \hat{\delta}_\xi g)$ is the symplectic potential 3-form. On the other hand, using the Lie derivative $\mathcal{L}_\xi$

$$\hat{\delta}_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = d(\xi \cdot \mathbf{L}). \tag{2}$$

Equating Eqs.(1) and (2), one gets

$$d * \mathbf{j}(\xi) + \mathbf{E}\hat{\delta}_\xi g = 0, \tag{3}$$

where

$$\mathbf{j}(\xi) = *(\mathbf{\Theta}(g, \hat{\delta}_\xi g) - \xi \cdot \mathbf{L}) \tag{4}$$

is the Noether current 1-form with respect to the diffeomorphism generated by a given vector field ξ^a. Its (entire) Noether charge is given by the integral over a Cauchy surface Σ

$$Q(\xi) = \int_\Sigma *\mathbf{j}(\xi). \tag{5}$$

In vacuum GR, the Lagrangian 4-form in units of $G = c = 1$ reads

$$\mathbf{L} = \frac{1}{16\pi} R\boldsymbol{\epsilon}, \tag{6}$$

where R is the scalar curvature and $\boldsymbol{\epsilon}$ the volume 4-form. The symplectic potential takes the form of

$$\boldsymbol{\Theta}_{abc}(g, \mathcal{L}_\xi g) = \frac{1}{16\pi}[\nabla^d(g_{ef}\hat{\delta}_\xi g^{ef}) - \nabla_e \hat{\delta}_\xi g^{de}]\boldsymbol{\epsilon}_{dabc}. \tag{7}$$

Thus, the Noether current and the Noether charge may be explicitly written as

$$\mathbf{j}_a(\xi) = \frac{1}{8\pi} G_{ab}\xi^b + \frac{1}{16\pi}(\nabla^b\nabla_a\xi_b - \nabla^b\nabla_b\xi_a) \tag{8}$$

and

$$Q(\xi) = \frac{1}{8\pi}\int_\Sigma *(G_{ab}\xi^b) - \frac{1}{16\pi}\int_{\partial\Sigma} *d\xi, \tag{9}$$

respectively, where G_{ab} is the Einstein tensor and $\partial\Sigma$ the boundary of the Cauchy surface. On shell, the first terms in Eqs. (8) and (9) vanish and the Noether charge may be expressed as an integral over the boundary of the Cauchy surface. The Noether current (8) is exactly the dual of the current 3-form constructed by Wald *et al* earlier (see, for example, [8, 9]). The integrand of the second term in (9) is nothing else but the Noether charge 2-form in [8, 9]. The boundary $\partial\Sigma$, in general, consists of two closed 2-dimensional surfaces at the two ends of the Cauchy surface. For the whole asymptotically flat region, the Cauchy surface emanates from the bifurcation surface and extends to the spatial infinity. Thus, the boundary of the Cauchy surface $\partial\Sigma$ should be $S_H^{(-)} \cup S_\infty$, where $S_H^{(-)}$ is the bifurcation surface. The superscript $(-)$ stands for the opposite orientation. Namely, its normal vector points to the direction of r decreasing. Then, the Noether charge becomes the algebraic summation of the partial Noether charges of the closed surfaces, i.e.

$$Q(\xi) = \mathcal{Q}_\infty(\xi) - \mathcal{Q}_H(\xi). \tag{10}$$

In particular, for the stationary, axisymmetric black hole space-time with Killing vector

$$\chi_K^a = t_K^a + \Omega_H \phi_K^a, \tag{11}$$

where t_K^a and ϕ_K^a are the time-like and space-like Killing vector of the space-time, respectively, Ω_H the angular velocity on the horizon,

$$\frac{1}{2}(\nabla_b\nabla^a\chi_K^b - \nabla_b\nabla^b\chi_K^a) = R_b^a\chi_K^b. \tag{12}$$

From Eq. (8), it follows that $\mathbf{j_a}$ and thus $Q(\chi_K)$ vanishes on shell. Since the mass and the angular momentum of the black hole are, by definition,

$$M = -\frac{1}{8\pi}\int_{S_\infty} *dt_K \tag{13}$$

and

$$J=\frac{1}{16\pi}\int_{S_\infty}*d\phi_K, \tag{14}$$

respectively, the expression for $\mathcal{Q}_\infty$ is then

$$\mathcal{Q}_\infty=-\frac{1}{16\pi}\int_{S_\infty}*d\chi_K=\frac{1}{2}M-\Omega_H J. \tag{15}$$

On the other hand, the expression for $\mathcal{Q}_H$ takes the form of

$$\mathcal{Q}_H=-\frac{1}{16\pi}\int_{S_H}*d\chi_K=\frac{\kappa}{8\pi}A, \tag{16}$$

where κ is the surface gravity and A the area of the cross section of the event horizon. Thus, the vanishing Noether charge $Q(\chi_K)=0$ gives rise to the mass formula [14]

$$Q=\frac{1}{2}M-\Omega_H J-\frac{\kappa}{8\pi}A=0. \tag{17}$$

The Noether charge may be defined for a finite region of a space-time. When Σ is not chosen to be the whole of the Cauchy surface but a partial Cauchy surface Σ_p with two boundaries B_1 and B_2 such that $A_{B_2}>A_{B_1}$, where A_{B_i} stands for the area of surface B_i,

$$Q_{\Sigma_p}(\xi)=\mathcal{Q}_{B_2}(\xi)-\mathcal{Q}_{B_1}(\xi) \tag{18}$$

gives the Noether charge for the portion of the space-time region $R\times\Sigma_p$.

3 Realization of Algebra *diff*$(\mathcal{M})$

In order to get the realization of algebra *diff*$(\mathcal{M})$, it is needed to consider the two successive horizontal variations of the Lagrangian 4-form induced by two vector fields ξ_1 and ξ_2. Due to the property

$$[\hat{\delta}_{\xi_1},\hat{\delta}_{\xi_2}]\mathbf{L}=\hat{\delta}_{[\xi_1,\xi_2]}\mathbf{L}, \tag{19}$$

it is straightforward to get

$$d\{\hat{\delta}_{\xi_1}[*\mathbf{j}(\xi_2)]-\hat{\delta}_{\xi_2}[*\mathbf{j}(\xi_1)]-*\mathbf{j}([\xi_1,\xi_2])\}=\hat{\delta}_{\xi_2}(\mathbf{E}\hat{\delta}_{\xi_1}g)-\hat{\delta}_{\xi_1}(\mathbf{E}\hat{\delta}_{\xi_2}g)+\mathbf{E}\hat{\delta}_{[\xi_1,\xi_2]}g. \tag{20}$$

Namely, the combination of the current 1-forms

$$*\hat{\delta}_{\xi_1}[*\mathbf{j}(\xi_2)]-*\hat{\delta}_{\xi_1}[*\mathbf{j}(\xi_2)]-\mathbf{j}([\xi_1,\xi_2]) \tag{21}$$

is also conserved on shell. The Noether-like charge $-K(\xi_1,\xi_2)$ with respect to this combination on a Cauchy surface Σ is then given by

$$-K(\xi_1,\xi_2)=\hat{\delta}_{\xi_1}Q(\xi_2)-\hat{\delta}_{\xi_2}Q(\xi_1)-Q([\xi_1,\xi_2]). \tag{22}$$

It should be noted that in Eq.(22) the (horizontal) variation $\hat{\delta}_{\xi_1}$ acts on both Q and ξ_2. Namely,

$$\hat{\delta}_{\xi_1} Q(\xi_2) = (\hat{\delta}_{\xi_1} Q)(\xi_2) + Q(\hat{\delta}_{\xi_1}\xi_2) = (\hat{\delta}_{\xi_1} Q)(\xi_2) + Q([\xi_1, \xi_2]). \tag{23}$$

Hence, Eq.(22) gives rise to

$$(\hat{\delta}_{\xi_2} Q)(\xi_1) - (\hat{\delta}_{\xi_1} Q)(\xi_2) = Q([\xi_1, \xi_2]) + K(\xi_1, \xi_2). \tag{24}$$

Eq.(24) shows that the Noether charges and their horizontal variations form an algebraic relation of *diff*$(\mathcal{M})$ and that $K(\xi_1, \xi_2)$ may be treated as the possible central term. (Note that in Eq.(24) the vector fields keep unchanged under the variation, i.e. $\hat{\delta}_{\xi_1}\xi_2 = \hat{\delta}_{\xi_2}\xi_1 = 0$ [8].)

Further, the Jacobi identity of the horizontal variations

$$([\hat{\delta}_{\xi_1}, [\hat{\delta}_{\xi_2}, \hat{\delta}_{\xi_3}]] + [\hat{\delta}_{\xi_2}, [\hat{\delta}_{\xi_3}, \hat{\delta}_{\xi_1}]] + [\hat{\delta}_{\xi_3}, [\hat{\delta}_{\xi_1}, \hat{\delta}_{\xi_2}]])\mathbf{L} = 0 \tag{25}$$

results in

$$K([\xi_1, \xi_2], \xi_3) + K([\xi_2, \xi_3], \xi_1) + K([\xi_3, \xi_1], \xi_2) = 0. \tag{26}$$

This is the two co-cycle condition for the center term.

To determine the possible central term, one has to calculate $(\hat{\delta}_{\xi_2} Q)(\xi_1) - (\hat{\delta}_{\xi_1} Q)(\xi_2)$ and $Q([\xi_1, \xi_2])$. By definition and the conservation equation $d * \mathbf{j} = 0$,

$$(\hat{\delta}_{\xi_2} Q)(\xi_1) - (\hat{\delta}_{\xi_1} Q)(\xi_2) = \int_{\partial\Sigma} \xi_2 \cdot \mathbf{\Theta}(g, \mathcal{L}_{\xi_1} g) - \xi_1 \cdot \mathbf{\Theta}(g, \mathcal{L}_{\xi_2} g) + \xi_1 \cdot (\xi_2 \cdot \mathbf{L}) - \xi_2 \cdot (\xi_1 \cdot \mathbf{L}). \tag{27}$$

For vacuum GR, the Lagrangian $\mathbf{L}$ vanishes on-shell and thus

$$(\hat{\delta}_{\xi_2} Q)(\xi_1) - (\hat{\delta}_{\xi_1} Q)(\xi_2) = \frac{1}{16\pi}\int_{\partial\Sigma} \boldsymbol{\epsilon}_{abcd}[\xi_2^c \nabla_e(\nabla^e \xi_1^d - \nabla^d \xi_1^e) - \xi_1^c \nabla_e(\nabla^e \xi_2^d - \nabla^d \xi_2^e)]. \tag{28}$$

On the other hand,

$$Q([\xi_1, \xi_2]) = -\frac{1}{16\pi}\int_{\partial\Sigma} \boldsymbol{\epsilon}_{abcd} \nabla^c (\xi_1^e \nabla_e \xi_2^d - \xi_2^e \nabla_e \xi_1^d). \tag{29}$$

The possible central term is then

$$\begin{aligned} K(\xi_1, \xi_2) &= (\hat{\delta}_{\xi_2} Q)(\xi_1) - (\hat{\delta}_{\xi_1} Q)(\xi_2) - Q([\xi_1, \xi_2]) \\ &= \frac{1}{8\pi}\int_{\partial\Sigma} \boldsymbol{\epsilon}_{abcd} \left[\nabla_e(\xi^c_{[1} \nabla^d \xi^e_{2]}) - 2\xi^{[c}_{[1} \nabla^{e]} \nabla_{|e|} \xi^d_{2]}\right]. \end{aligned} \tag{30}$$

The above analysis may also apply to a hyperbolic region with a partial Cauchy surface Σ_p in the manifold. Namely, the Noether charges $Q_{\Sigma_p}(\xi)$s for the hyperbolic region with a partial Cauchy surface and their horizontal variation form the algebraic relation of *diff*$(R \times \Sigma_p)$,

$$(\hat{\delta}_{\xi_2} Q_{\Sigma_p})(\xi_1) - (\hat{\delta}_{\xi_1} Q_{\Sigma_p})(\xi_2) = Q_{\Sigma_p}([\xi_1, \xi_2]) + K_{\Sigma_p}(\xi_1, \xi_2). \tag{31}$$

Since Q_{Σ_p} is expressed on shell in terms of the algebraic summation of the boundary terms as Eq. (18), Eq. (31) can be separated into two algebraic relations

$$(\hat{\delta}_{\xi_2} \mathcal{Q}_{B_i})(\xi_1) - (\hat{\delta}_{\xi_1} \mathcal{Q}_{B_i})(\xi_2) = \mathcal{Q}_{B_i}([\xi_1, \xi_2]) + K_{B_i}(\xi_1, \xi_2), \qquad i = 1, 2 \tag{32}$$

with help of Eq. (18).

4 Reduced algebra on event horizon of stationary axisymmetric black hole

Let us consider such a partial Cauchy surface Σ_1 that it emanates from the bifurcation surface S_H, extends almost along the generator of event horizon of a black hole and ends at certain place of the stretched Killing horizon [11] denoted by B_ϵ. At the end of calculation, this partial Cauchy surface tends to the event horizon by taking $\epsilon \to 0$. Define a vector field orthogonal to the Killing vector fields (11) by

$$\nabla_a \chi_K^2 = -2\kappa\rho_a, \tag{33}$$

where κ is the surface gravity on the event horizon. The two vector fields χ_K^a and ρ^a may combine into the two null vectors,

$$\begin{aligned} l^a &= \frac{1}{2}(\chi_K^a + \frac{|\chi_K|}{\rho}\rho^a) \\ n^a &= -\frac{1}{\chi_K^2}(\chi_K^a - \frac{|\chi_K|}{\rho}\rho^a). \end{aligned} \tag{34}$$

l^a, n^a and $m^a, \bar{m}^a$ constitue a null tetrad field in the neigborhood of the event horizon. The Lie bracket of l^a and n^a reads

$$[l, n]^a = -\kappa\frac{\rho}{|\chi|}n^a. \tag{35}$$

It may also be checked that

$$\begin{aligned} D\chi_K^2 &:= l^a\nabla_a\chi_K^2 = O(\chi_K^2) \\ \Delta\chi_K^2 &:= n^a\nabla_a\chi_K^2 = O(1). \end{aligned} \tag{36}$$

Therefore, the vector fields of type

$$\xi^a = Tl^a + Rn^a \qquad \text{with } R \sim O(\chi_K^2) \tag{37}$$

form a closed algebra under the Lie bracket.

For this type of vector fields, the partial Noether charge $\mathcal{Q}$ reads

$$\mathcal{Q}_S(\xi) = -\frac{1}{16\pi}\int_S \hat{\boldsymbol{\epsilon}}_{ab}(DT - \Delta R + \kappa T) \tag{38}$$

and Eq. (28) for the partial Noether charge $\mathcal{Q}$ reduces to

$$(\hat{\delta}_{\xi_2}\mathcal{Q}_S)(\xi_1) - (\hat{\delta}_{\xi_1}\mathcal{Q}_S)(\xi_2) = \frac{1}{8\pi}\int_S \hat{\boldsymbol{\epsilon}}_{ab}(T_{[1}D\Delta R_{2]} - T_{[1}D^2T_{2]} - \kappa T_{[1}DT_{2]}) \tag{39}$$

when the boundary condition $l_{a;\,b}(m^a\bar{m}^b + \bar{m}^a m^b)|_S = 0$ is satisfied, where $S = S_H$ or $\lim\limits_{\epsilon\to 0} B_\epsilon$ (denoted by B hereafter), and $\hat{\boldsymbol{\epsilon}}_{ab}$ is the area 2-form of S. The straightforward calculation shows

$$\mathcal{Q}_S([\xi_1, \xi_2]) = -\frac{1}{8\pi}\int_S \hat{\boldsymbol{\epsilon}}_{ab}(-T_{[1}D\Delta R_{2]} + T_{[1}D^2T_{2]} + \kappa T_{[1}DT_{2]}). \tag{40}$$

Thus, the central term vanishes!

5 Concluding Remarks

In conclusion, as other symmetries in other classical and quantum field theories, the diffeomorphism algebra, reflecting the diffeomorphism invariance of diffeomorphism invariant theories, may be realized by the Noether charges and their horizontal variations. The Noether-charge realization has four remarkable features. First of all, the Noether-charge realization always exists because the Noether charge always exists for any given vector fields and any given boundary conditions. This is in contrast to the Hamiltonian-functional realization, which does not exist for some vector fields and boundary conditions because the Hamiltonian functionals themselves do not always exist [8]. Secondly, the Noether-charge realization is a completely covariant approach. In the present approach, the Poisson bracket and Dirac bracket are not used at all, which are defined in the canonical approach. Thirdly, only the horizontal variations are considered in the Noether-charge realization. Finally, for vacuum general relativity the Noether-charge realization has the same form as the one given by Carlip with the help of the Hamiltonian functionals [11].

Another conclusion of the present letter is that the central term on Killing horizon for a large class of vector fields vanishes! The key point is that the null tetrad instead of the basis $\{\chi_K^a, \rho^a, t_1^a, t_2^a\}$ is used. The former is well defined everywhere, including on the Killing horizon, while the latter is ill-defined on the Killing horizon. Therefore, the appearance of the central term seems to come from the choice of basis.

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